

ARAKELOV-PARSHIN RIGIDITY OF TOWERS OF CURVE FIBRATIONS, CONNECTIONS TO THE INFINITESIMAL TORELLI PROBLEM

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ABSTRACT. The question of higher dimensional Arakelov-Parshin rigidity asks when it is impossible to deform families of canonically polarized manifolds without changing their base. It is one of the three main pieces in higher dimensional Shafarevich conjecture. By a different, already proven piece, for any class of families with fixed Hilbert polynomial, Arakelov-Parshin rigidity yields finiteness of the given class.

In the present article rigidity of towers of smooth curve fibrations with genera at least two is examined. In the compact base case, it turns out that, apart from an obvious exception, if any variation is zero, then some cover of the tower can be deformed. In the meanwhile if all variations are non-zero, then the tower is rigid. The arbitrary base case is much more obscure. In that case rigidity is proven for level two towers with maximal variations. The method used there is showing that the iterated Kodaira-Spencer map is injective. In the end this method is related to the infinitesimal Torelli problem. It is shown that if the multiplication map from canonical to bicanonical sections is surjective, then the injectivity of the iterated Kodaira-Spencer map implies the injectivity of the tangent map of the period map.

1. INTRODUCTION

According to Grothendieck's functor of points point of view, a way to understand a space is to understand maps into it from all schemes. This is specially true, if no concrete description is available, only some properties of a space are known. An example for that is the moduli stack \mathcal{M}_g of smooth curves of genus g , for $g \geq 2$. Over the complex numbers, the first interesting class of maps to \mathcal{M}_g are finite morphisms from curves. There is an intriguing classical result, one of the famous conjectures of Shafarevich about this class. To state it, fix an integer $g \geq 2$, a smooth (not necessarily projective) curve U , its compactification B and define $\Delta := B \setminus U$. Call a smooth family isotrivial, if all its fibers are isomorphic.

Theorem 1.1 Shafarevich Conjecture, [Par68], [Ara71].

(1.1.1) ***Finiteness (F):** There are finitely many non-isotrivial families of smooth projective curves of genus g over U .*

(1.1.2) **Hyperbolicity (H):** *If $2g(B) - 2 + \#\Delta \leq 0$, then there are no such families.*

Evenmore, since it will play an important role later on, we state how finiteness was decomposed to two statements in the original proofs of Arakelov and Parshin.

Theorem 1.2 Finiteness part of Shafarevich conjecture, [Par68], [Ara71].

(1.2.1) **Boundedness (B):** *There are finitely many deformation types of non-isotrivial smooth families of curves of genus g over U .*

(1.2.2) **Rigidity (R):** *Every non-isotrivial family of smooth curves of genus g over U is rigid. That is, its deformation type contains only one element.*

In the last two decades there has been an enormous progress in generalizing these, by now, classical statements to higher dimensions. In the generalizations, first \mathcal{M}_g is replaced by its higher dimensional generalization, the moduli space of canonically polarized manifolds \mathcal{M}_h with fixed Hilbert polynomial h ([Vie95]). We note that the compactification of the latter moduli space is an exciting ongoing project (e.g., [HK10], [Kol]). In the most general form of higher dimensional Shafarevich conjecture, after replacing \mathcal{M}_g by \mathcal{M}_h , usually arbitrary dimensional bases are allowed. The main subject of the present article is the generalizations of **(R)**. However, we give a short account of the other generalizations too. First in Conjecture 1.3 the expectations are summarized. Then we give a brief summary of the available results.

To state Conjecture 1.3, we generalize our earlier notations. From now, let U be manifold (i.e. smooth variety), B a smooth compactification of U such that $\Delta := B \setminus U$ is a global normal crossing divisor. Fix also a polynomial h . The variation $\text{Var } f$ of a family $f : X \rightarrow U$ of canonically polarized manifolds with Hilbert polynomial h is $\dim(\text{im } \nu)$, where $\nu : U \rightarrow \mathcal{M}_h$ is the moduli map.

Conjecture 1.3 Higher dimensional version of Shafarevich conjecture.

(1.3.1) **(B):** *Families of canonically polarized manifolds over U with Hilbert polynomial h fall into finitely many deformation equivalence classes.*

(1.3.2) **(R):** *No good comprehensive conjecture is known (See Question 1.5). However there is **Viehweg's rigidity conjecture**: If $f : X \rightarrow U$ is a family of projective manifolds with $\Omega_{X/U}$ relatively ample, then f is rigid.*

(1.3.3) **(H) :** *There are multiple conjectures concerning Hyperbolicity. Consider a family $f : X \rightarrow U$ of canonically polarized manifolds.*

- (a) **Viehweg's hyperbolicity conjecture:** If $\text{Var } f = \dim B$, then $\omega_B(\Delta)$ is big (or with other words $\kappa(B, \Delta) = \dim B$).
- (b) **Kebekus-Kovács conjecture:**
 - (i) If $\kappa(B, \Delta) < -\infty$ and $\text{Var } f < \dim Y$.
 - (ii) If $\kappa(B, \Delta) \geq 0$ and $\text{Var } f \leq k(B, \Delta)$.
- (c) **Campana conjecture:** If (B, Δ) is special, then f is isotrivial (Special means that for every p and every line bundle $\mathcal{L} \subseteq \Omega_B^p(\log \Delta)$, $k(\mathcal{L}) < p$).

REMARK 1.4. Kebekus-Kovács conjecture is a generalization of Viehweg's hyperbolicity conjecture, and reflects the conjectured birational coincidence of the moduli map $U \rightarrow \mathcal{M}_h$ and of the fibrations given by the log minimal model of (B, Δ) .

The biggest success has been undoubtedly **(B)**. First some weak form of boundedness was proven in the $\dim B = 1$ case (i.e. that $f_*\omega_{X/B}^m$ is bounded in terms of $g(B)$, $\#\Delta$, h and m). This was done in [BV00] and then generalized to mildly singular fibers in [VZ01] and [Kov02]. Then in [KL06] boundedness was shown for arbitrary base and fiber dimensions.

By now **(H)** is also on a good track to be completed. However its story included lot more chapters. The first portion of results were about the $\dim B = 1$ case. That case was proven first in [Mig95] for $\Delta = \emptyset$ and $\dim F = 2$ (where F is the general fiber of f). Then it was extended in [Kov96] to arbitrary fiber dimensions. The $\Delta \neq \emptyset$ case was proven first for $\dim F = 2$ ([Kov97]). Finally the general statement of **(H)** for one dimensional base was proven in [Kov00].

For the arbitrary $\dim B$ case, Viehweg's hyperbolicity Conjecture holds when B is a projective space or a hyperquadric ([VZ02], [Kov03b]), In [VZ02] it was also shown for semi-positive $\mathcal{T}_B^1(\log \Delta)$, and various complete intersections in \mathbb{P}^n . Then in [Kov03a] it was proven for uniruled base with Picard number 1. The $\dim B = 2$ case was entirely settled in [KK08a] and $\dim B = 3$ in [KK]. Evenmore in [KK], the Kebekus-Kovács conjecture was proven for bases of dimension at most three. The $\Delta = \emptyset$ case has been also shown for arbitrary dimension of the base, assuming that minimal model program works [KK08b]. Campana's conjecture was established for $\dim B \leq 3$ in [JKb] and [JKa].

In contrast to the spectacular results in **(B)**, **(H)**, there is little known about **(R)**, although it was in the focus of the same researchers as the other two. The basic reason for that is that when we have higher dimensional fibers, then $\text{Var } f = \dim B$ is not enough assumption to obtain rigidity (see Example 2.2). Loosely speaking some strong hyperbolicity or variational

assumption (or both) is needed to obtain rigid families. So, the higher dimensional version of **(R)** is more of a question so far, and is as follows. For the precise definition of rigid families consult Definition 2.1.

QUESTION 1.5. Which families $f : X \rightarrow B$ of canonically polarized manifolds are rigid? Evenmore we are really interested in rigid families f , all coverings f' of all quasi-finite pullbacks of which are still rigid.

$$\begin{array}{ccccc}
 X & \xleftarrow{\quad} & X \times_{B'} X & \xleftarrow{\text{quasi-finite}} & X' \\
 f \downarrow \text{rigid} & \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow & \downarrow \text{rigid} & & \swarrow f', \text{ family of canonically polarized manifolds} \\
 B & \xleftarrow{\text{quasi-finite}} & B' & \xleftarrow{\text{manifold}} &
 \end{array}$$

We call such families stably rigid.

REMARK 1.6. We added the note about stable rigidity, because it seems to capture the philosophy of the rigidity condition of Theorem 1.2 (i.e. of the original Shafarevich conjecture). More precisely, the stably rigid families of smooth curves of genus at least two are exactly the non-isotrivial families by Proposition 2.3. We are expecting same stability properties for any good rigidity condition in higher dimensions.

Whatever answer one gives for Question 1.5, it yields **(F)** by **(B)**. More precisely [KL06, Theorem 1.6] implies the following theorem.

Theorem 1.7. *If for a fixed manifold B and polynomial h , \mathcal{C} is a class of rigid families of canonically polarized projective manifolds with Hilbert polynomial h , then \mathcal{C} is finite.*

So, far there has been one answer to Question 1.5. For a family $f : X \rightarrow B$ of relative dimension n , in Definition 2.4 the iterated Kodaira-Spencer map $\text{iks}_f : S^n(\mathcal{T}_B) \rightarrow R^n f_*(\wedge^n \mathcal{T}_{X/B})$ is defined. Its definition is motivated by Hodge Theory, and in case $n = 1$ it specializes to the ordinary Kodaira-Spencer map. It is interesting for us, because its injectivity implies rigidity by [VZ03, Corollary 8.4] or [Kov05, Theorem 4.14].

Theorem 1.8. *If $f : X \rightarrow B$ is a family of canonically polarized manifolds over a smooth (not necessarily projective) curve, such that iks_f is injective, then f is rigid.*

The problem with iterated Kodaira-Spencer map is that we have no geometric understanding of it unless $n = 1$ or if f is a family of hypersurfaces (see [VZ05] for the latter case). Being a notion motivated by Hodge Theory, its understanding is equivalent to understanding certain aspects of the Torelli map of some Hodge structures. See Theorem 1.20 for some details on this. Unfortunately the Torelli map of canonically polarized manifolds is very hard to understand for higher dimensions. So, this connection might

indicate that there are easier ways to tackle Question 1.5 than computing the iterated Kodaira-Spencer map of a family. It also explains why the hypersurface case is the only one which is understood, since Hodge structures of hypersurfaces have very good descriptions.

1.A. Results of the paper

As we have seen, there are no known answers to Question 1.5 for higher dimensional fibers, apart from the hypersurface case. In the present article we start filling in this gap. We analyze the rigidity of towers of curve fibrations. More precisely we consider the following situation.

NOTATION 1.9. A tower of curve fibrations is a morphism $f : X \rightarrow B$ fitting in a commutative diagram

$$\begin{array}{c} \xrightarrow{\quad f \quad} \\ X = X_n \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 = B, \end{array}$$

where all schemes are varieties and the generic fibers of all f_i are one dimensional and connected. In the present article we examine rigidity of towers of curve fibrations where B is furthermore a smooth curve (not necessarily projective), and f_i are families of smooth curves with genus at least two.

REMARK 1.10. Note, that, using Notation 1.9, if all f_i are families of smooth curves of genera at least two, then X_b is canonically polarized for all $b \in B$ by Proposition 3.1.

MOTIVATION 1.11. Considering towers of curve fibrations is motivated partially by the following fact, which states that all families can be approximated in certain senses with towers of curve fibrations. Hence, we hope that in the long run, results about towers can be extended to general families.

By [dJ97, Corollary 5.10] every family $g : Y \rightarrow Z$ can be altered to a tower of curve fibrations such that f_i are semi-stable families of curves. This means that there is a commutative diagram

$$\begin{array}{ccc} Y & \xleftarrow{\text{generically finite, proper}} & X \\ \downarrow g & & \downarrow f \\ Z & \xleftarrow{\text{generically finite, proper}} & B, \end{array}$$

with f such a tower. This fact would be even more promising if the answer to, the deliberately vaguely worded, Question 1.12 was yes. It would mean, that every non-rigid family could be altered to a non-rigid tower of curve

fibrations (using [HM06, Corollary 1.4]). Hence stably rigid families could be determined by examining towers of curve fibrations only.

QUESTION 1.12. If $g : W \rightarrow Z \times T$ is a deformation of the family $g_0 : W_{t_0} \rightarrow Z \times \{t_0\}$ of canonically polarized manifolds, is there then an alteration of the deformation g into a deformation of a tower of curve fibrations?

REMARK 1.13. Since Question 1.12 is only a motivation for the results of the paper, we do not try to answer it here. Certainly the answer is yes, if the conditions are relaxed enough (e.g., non-irreducible X_i and non-irreducible fibers for f_i are allowed). So, the question is more, with which conditions is the answer yes.

MOTIVATION 1.14. Another motivation to examine rigidity in the situation of Notation 1.9 is that if $n = 2$ and $\text{Var } f_i = i$, it gives a special case of Viehweg's rigidity conjecture (e.g., [Sch86, Theorem 2]).

Now, we state the results of the paper. First for the case of compact B , we have an almost full characterization of stable rigidity. Unfortunately, there is one possibility, which obstructs giving a very short answer, which will be explained after the statement of the theorem.

Theorem 1.15. *In the situation of Notation 1.9, if B is projective, then*

- (1.15.1) *if $\text{Var } f_i \geq 1$ for all i then f is rigid and*
- (1.15.2) *otherwise there is a commutative diagram*

$$\begin{array}{ccc} X & \xleftarrow{\text{\textit{\'etale}}} & X' \cong W \times Y \\ \downarrow f & & \downarrow f' \\ B & \xleftarrow{\text{\textit{\'etale, finite}}} & B' \end{array}$$

where $W \rightarrow B'$ is a family of canonically polarized manifolds, and Y is a positive dimensional canonically polarized manifold (the map $W \times Y \rightarrow B'$ is the first projection composed with $W \rightarrow B'$). In particular, if Y is not a rigid manifold, then f' is not rigid.

REMARK 1.16. The aforementioned possibility which obstructs a perfect characterization of stable rigidity is the case when Y is a rigid manifold. Unfortunately all we know about Y is that it is a tower of curve fibrations. However, that can still be rigid. In fact if one has a tower of curve fibrations which is rigid there is an immediate example of f for which not all $\text{Var } f_i \geq 1$, but it is rigid. One just takes another tower $W \rightarrow B$ where all variations are at least one, and then $W \times Y \rightarrow B$ is rigid by the computations of Section 4.

In the arbitrary base case, for two level towers with maximal variations we can prove the injectivity of the iterated Kodaira-Spencer map.

Theorem 1.17. *In the situation of Notation 1.9, if $n = 2$ and $\text{Var } f_i = i$ (i.e. variations are maximal), then iks_f is injective (see Definition 2.4 for the definition of iks_f). In particular, such families are rigid by Theorem 1.8.*

REMARK 1.18. A smooth non-isotrivial morphism $Y \rightarrow C$ is called a Kodaira fibration if both the fibers and the base are smooth projective curves of genera at least two. Consider a deformation $W \rightarrow Z$ of a surface Y admitting a Kodaira fibration $Y \rightarrow C$. Then, after an étale base change $W \rightarrow Z$ becomes a family of Kodaira fibrations. That is, one can find an étale map $Z' \rightarrow Z$ such that there is a commutative diagram

$$\begin{array}{ccccc} & & \text{---} \curvearrowright \text{---} & & \\ & & \text{---} & & \\ W \times_Z Z' & \longrightarrow & S & \longrightarrow & Z', \end{array}$$

where all restrictions of the diagram over $z \in Z'$ give Kodaira fibrations. Hence Theorem 1.17 can be interpreted as a rigidity criteria for surfaces admitting Kodaira fibrations. Since it is not the focus of the article, we do not prove the statements of the remark here.

An immediate corollary of Theorem 1.15, Theorem 1.17 and Theorem 1.7 is a finiteness statement.

Corollary 1.19. *Fixing a smooth curve B and a polynomial h , there are finitely many*

- towers $f : X \rightarrow B$ as in Notation 1.9 with $\text{Var } f_i \geq 1$, projective B and Hilbert-polynomial h and
- towers $f : X \rightarrow B$ as in Notation 1.9 with $\text{Var } f_i = i$, $n = 2$ and Hilbert-polynomial h .

After obtaining our rigidity results we show a connection between the infinitesimal Torelli problem and the injectivity of the Kodaira-Spencer map. See Section 6 for the notation and for a short overview on the infinitesimal Torelli problem.

Theorem 1.20. *In the situation of Notation 1.9, with $n = 2$, and $\text{Var } f_i = i$, such that for some $b \in B$, $S^2(H^0(X_b, \omega_{X_b})) \rightarrow H^0(X_b, \omega_{X_b}^{\otimes 2})$ is surjective, the tangent map $T\phi_f$ of the period map is injective. That is, for generic $b \in B$, the infinitesimal Torelli problem holds at $[X_b]$ in the direction defined by B .*

1.B. Organization of the paper

The following two sections are preparations for the follow-ups. In section 2, the introductory definitions and statements left out from Section 1, to avoid technicalities there, are collected. Section 3 is a short account on the results

used in the paper about the positivity of the relative canonical sheaves. Then in Section 4, Theorem 1.15 is proven. Along doing so, some facts about the moduli theory of families of canonically polarized manifolds is collected. Section 5 is entirely devoted to the proof of Theorem 1.17. Then in the end in Section 6, the link between the injectivity of the iterated Kodaira-Spencer map and the infinitesimal Torelli problem is presented.

1.C. Notation

We work over an algebraically closed field k of characteristic zero. However, sometimes (e.g., Section 6) we have to assume that the base field is \mathbb{C} . All schemes are of finite type over k unless otherwise stated. For a curve C , $g(C)$ denotes its genus. A manifold is a smooth variety. A variety is an integral, separated scheme of finite type over k . A global normal crossing divisor is defined Zariski locally by $\prod f_i^{n_i}$ where f_i are regular elements and n_i are positive integers. A canonically polarized manifold is a projective manifold Z with ample ω_Z . The Hilbert polynomial of a canonically polarized manifold Z is $h(n) := \chi(\omega_Z^n)$. The Kodaira and log Kodaira dimensions of a variety Z or a pair (Z, Δ) is denoted by $\kappa(Z)$ and $\kappa(Z, \Delta)$, respectively. For a line bundle \mathcal{L} , its Iitaka-Kodaira dimension is denoted by $\kappa(\mathcal{L})$. We say, the variation of a family $g : Y \rightarrow Z$ is maximal if $\text{Var } g = \dim Z$. A vector bundle \mathcal{E} on Y is ample over an open set U , if there is an ample line bundle \mathcal{L} and a homomorphism $\mathcal{L}^{\oplus N} \rightarrow \mathcal{E}$, which is surjection over U . \mathcal{E} is ample if it is ample over X . We denote by \mathcal{M}_g and \mathcal{M}_h the moduli stacks of smooth projective curves of genus g and canonically polarized manifolds of Hilbert-polynomial h , respectively.

2. BASIC CONCEPTS

Here we collected some basic definitions and constructions mentioned in Section 1, which being slightly technical were omitted from there. We start with the precise definition of rigidity.

Definition 2.1. A family $X \rightarrow B$ of canonically polarized manifolds is *rigid*, if for every deformation of f

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow f & & \downarrow f' \\ B & \longrightarrow & B \times S \end{array}$$

over a smooth curve S , to families of canonically polarized manifolds there is an isomorphism for all $s \in S$

$$\begin{array}{ccc} X & \xrightarrow{\cong} & (X')_s \\ \downarrow f & & \downarrow f' \\ B & \xrightarrow{x \mapsto (x,s)} & B \times \{s\}. \end{array}$$

Next we show the promised example about why maximal variation does not imply rigidity for higher dimensional fibers.

EXAMPLE 2.2. Consider two non-isotrivial families $f : S \rightarrow C$ and $g : T \rightarrow D$ of smooth curves of genus at least two the bases of which are also curves of genera at least two. Such families exist (e.g., [BPVdV84, Section V.14]). Consider $f \times g : S \times T \rightarrow C \times D$. It is a family of canonically polarized surfaces over $C \times D$. Moreover, since by [HM06, Corollary 1.4], from a fixed variety there are only finitely many dominant maps onto varieties of general type up to birational equivalence, the restriction of $S \times T \rightarrow C \times D$ to $\{c\} \times D$ or $C \times \{d\}$ is non-isotrivial for any $c \in C$ and $d \in D$. So, fix any $d \in D$. Then $S \times T \rightarrow C \times D$ is a non-trivial deformation of the non-isotrivial family $S \times T_d \rightarrow C \times \{d\} \cong C$ of canonically polarized manifolds. However, since $\dim C = 1$ here non-isotrivial means having maximal variation. So, maximal variation does not imply rigidity in case of higher dimensional fibers.

The next proposition was promised after the statement of Question 1.5 and justifies the introduction of stable rigid families.

Proposition 2.3. *A family $f : X \rightarrow B$ of smooth curves of genus at least two is stably rigid, if and only if it is non-isotrivial.*

Proof. From Theorem 1.2, using that non-isotriviality is stable under pulling back and taking cover ([HM06, Corollary 1.4]), follows the backwards direction. To see the forward direction, assume that $f : X \rightarrow B$ is isotrivial. Then by Lemma 4.4 (with setting $Y := X$, $T := B$, $S := \operatorname{Spec} k$), it follows, that there is a finite étale cover $U \rightarrow B$, such that there is a diagram

$$\begin{array}{ccc} X \times_B U & \xrightarrow{\cong} & U \times F \\ \downarrow & & \downarrow \\ U & \xlongequal{\quad} & U \end{array}$$

for some smooth curve F of genus at least two. Then by deforming F , we get a deformation of $X \times_B U \rightarrow U$. That is, f is not stably rigid. \square

The rest of the section is devoted to the definition of the iterated Kodaira Spencer map. It is the main object of Sections 5 and 6.

Definition 2.4. If $g : Y \rightarrow Z$ is a proper, smooth morphism of relative dimension n over a smooth base, then for $1 \leq p \leq n$, by [Har77, Exercise II.5.16] $\wedge^p \mathcal{T}_Y$ has a filtration $0 = \mathcal{F}_0^p \subseteq \mathcal{F}_1^p \subseteq \cdots \subseteq \mathcal{F}_p^p \subseteq \mathcal{F}_{p+1}^p = \wedge^p \mathcal{T}_Y$ by locally free sheaves such that the induced quotients are

$$\mathcal{F}_{i+1}^p / \mathcal{F}_i^p \cong (g^* \wedge^i \mathcal{T}_Z) \otimes (\wedge^{p-i} \mathcal{T}_{Y/Z})$$

Consider then the short exact sequences

$$0 \longrightarrow \wedge^p \mathcal{T}_{Y/Z} \longrightarrow \mathcal{F}_2^p \longrightarrow g^* \mathcal{T}_Z \otimes \wedge^{p-1} \mathcal{T}_{Y/Z} \longrightarrow 0.$$

Tensor these with $g^* \mathcal{T}_Z^{\otimes n-p}$ to get the exact sequences (2.4.1)

$$\begin{aligned} 0 \longrightarrow g^* \mathcal{T}_Z^{\otimes n-p} \otimes \wedge^p \mathcal{T}_{Y/Z} &\longrightarrow g^* \mathcal{T}_Z^{\otimes n-p} \otimes \mathcal{F}_2^p \\ &\longrightarrow g^* \mathcal{T}_Z^{\otimes n-p+1} \otimes \wedge^{p-1} \mathcal{T}_{Y/Z} \longrightarrow 0. \end{aligned}$$

Denote by ρ_p the edge maps

$$\rho_p : \mathcal{T}_Z^{\otimes(n-p+1)} \otimes R^{p-1} g_*(\wedge^{p-1} \mathcal{T}_{Y/Z}) \rightarrow \mathcal{T}_Z^{\otimes(n-p)} \otimes R^p g_*(\wedge^p \mathcal{T}_{Y/Z})$$

obtained by applying higher pushforwards to (2.4.1). Then the Kodaira-Spencer map

$$\text{ks}_g : \mathcal{T}_Z \rightarrow R^1 g_* \mathcal{T}_{Y/Z}$$

of g is $\rho_1 \otimes \text{id}_{\mathcal{T}_Z^{-(n-1)}}$ and define the iterated Kodaira-Spencer map

$$\text{iks}_g : \mathcal{T}_Z^n \rightarrow R^n g_*(\wedge^n \mathcal{T}_{Y/Z})$$

of g to be $\rho_n \circ \cdots \circ \rho_1$. We also define the i -th iterated Kodaira-Spencer map

$$\text{iks}_g^i : \mathcal{T}_Z^n \rightarrow \mathcal{T}_Z^{n-i} \otimes R^i g_*(\wedge^i \mathcal{T}_{Y/Z})$$

by $\rho_i \circ \cdots \circ \rho_1$

REMARK 2.5. In the case when $\dim Z = 1$, $\mathcal{F}_2^p = \wedge^p \mathcal{T}_Y$.

REMARK 2.6. There is another way to define iks_g . It is the composition of the n times product of ks_g and of the wedge product:

$$\begin{array}{ccc} & \text{iks}_g & \\ & \curvearrowright & \\ \mathcal{T}_Z^{\otimes n} & \xrightarrow{\text{ks}_f \otimes \cdots \otimes \text{ks}_f} (R^1 g_* \mathcal{T}_{Y/Z})^{\otimes n} & \xrightarrow{\wedge} R^n g_*(\wedge^n \mathcal{T}_{Y/Z}) \end{array}$$

The equivalence of the two definitions can be proven using Dolbeault cohomology.

3. POSITIVITY PROPERTIES OF THE RELATIVE CANONICAL SHEAF

In this section some positivity results are collected, some of which have already been used, and others will be used frequently later on. First, a statement about the relative canonical sheaves of a family of canonically polarized manifolds.

Proposition 3.1. *If $f : X \rightarrow B$ is a family of canonically polarized manifolds with B smooth, projective, then $\omega_{X/B}$ is nef.*

Proof. It is known that $f_*\omega_{X/B}$ is nef (e.g., [Vie83, Theorem 4.1]). Then since $\omega_{X/B}$ is relatively ample, there is some $n > 0$ such that $\omega_{X/B}^{\otimes n}$ is relatively globally generated. That is, there is a surjection $f^*f_*(\omega_{X/B}^{\otimes n}) \rightarrow \omega_{X/B}^{\otimes n}$, which shows the nefness of $\omega_{X/B}$. \square

Next, another statement about the pushforwards of tensor powers of the relative canonical sheaf (e.g., [VZ02, Proposition 3.4]).

Lemma 3.2. *If $f : X \rightarrow B$ is a family of canonically polarized manifolds with B smooth, projective and $\text{Var } f = \dim B$, then for any $\nu > 1$, $f_*(\omega_{X/B}^\nu)$ is ample with respect to the open subset $U \subseteq B$, where the moduli map $B \rightarrow \mathcal{M}_h$ is quasi-finite.*

Corollary 3.3. *In the situation of Lemma 3.2, $\omega_{X/B}$ is ample with respect to $f^{-1}U$.*

Proof. Since $\omega_{X/B}$ is relatively ample, $\omega_{X/B}^n$ is relatively globally generated for $n \gg 0$. Choose such an n . Then there is a surjection

$$\omega_{X/B} \otimes f^*f_*(\omega_{X/B}^n) \rightarrow \omega_{X/B} \otimes \omega_{X/B}^n \cong \omega_{X/B}^{n+1},$$

which yields the statement of the lemma using Proposition 3.1 and that relatively ample nef line bundle tensored with the pullback of an ample vector bundle over U is ample over $f^{-1}U$. \square

Corollary 3.4. *If $f : X \rightarrow B$ is a family of canonically polarized manifolds with B smooth, projective and with relative dimension n , then $\kappa(\omega_{X/B}) = \text{Var } f + n$.*

Proof. Let $\nu : B \rightarrow \mathcal{M}_h$ be the moduli map. One can construct a commutative diagram, all “vertical” squares of which are Cartesian

$$\begin{array}{ccccc}
 & \tilde{X} & \xrightarrow{\zeta} & Y & \\
 & \swarrow \xi & & \swarrow & \\
 X & \xrightarrow{\tilde{f}} & \mathcal{U}_h & & \\
 \downarrow f & & \downarrow & & \downarrow \\
 & \tilde{B} & \xrightarrow[\eta]{\text{surjective}} & D & \\
 \text{quasi-finite, surjective} \nearrow & & \downarrow \psi & & \downarrow \\
 B & \xrightarrow{\nu} & \mathcal{M}_h & &
 \end{array}$$

where \mathcal{U}_h is the universal family over \mathcal{M}_h and D is a smooth, proper scheme [Vie95, Theorem 9.25]. Since all vertical maps are smooth, all relative canonical sheaves are compatible with pullbacks. By Corollary 3.3, $\omega_{Y/D}$ is big. Hence

$$\begin{aligned}
 \kappa(\omega_{X/B}) &= \kappa(\xi^* \omega_{X/B}) = \kappa(\omega_{\tilde{X}/\tilde{B}}) = \kappa(\zeta^* \omega_{Y/D}) \\
 &= \kappa(\omega_{Y/D}) = \dim Y = \dim D + 1 = \text{Var } \tilde{f} + 1 = \text{Var } f + 1
 \end{aligned}$$

□

4. COMPACT BASES

In this section the compact base case (i.e. Theorem 1.15) is treated. Having a projective base allows us to use certain techniques not available in the general case. More precisely, the set of families of canonically polarized manifolds with fixed Hilbert polynomial form a nice moduli space if B is projective. This is worded by the following lemma. However, first some preparation is necessary.

Fix a projective manifold B , and a polynomial h . One can define a moduli functor $\mathcal{M}_{B,h}$ of families of canonically polarized manifolds with Hilbert polynomial h by

$$\mathcal{M}_{B,h}(T) := \left\{ f : X \rightarrow B \times T \left| \begin{array}{l} f \text{ is a smooth morphism, } \omega_f \text{ is } f\text{-} \\ \text{ample, and } \chi(\omega_f^n|_{X(b,t)}) = h(n) \text{ for} \\ \text{every } n \in \mathbb{Z} \text{ and } (b,t) \in B \times T \end{array} \right. \right\}$$

One can also give a natural category fibered in groupoid structure to this functor which we also denote by $\mathcal{M}_{B,h}$. Then the following lemma holds.

Lemma 4.1. $\mathcal{M}_{B,h} \cong \underline{\text{Hom}}(B, \mathcal{M}_h)$ as categories fibered in groupoids, where $\underline{\text{Hom}}(B, \mathcal{M}_h)$ is the Hom-stack ([Ols06b, Lines 1-4]). In particular

by [Ols06b, Theorem 1.1], $\mathcal{M}_{B,h}$ is a Deligne-Mumford stack, locally of finite type.

The next corollary is the reason why a locally of finite type DM stack structure on $\mathcal{M}_{B,h}$ is useful.

Corollary 4.2. *If $f : X \rightarrow B$ is a family of canonically polarized manifolds with Hilbert polynomial h over compact B , then f is rigid (according to Definition 2.1) if the infinitesimal deformation space $T^1(f, \mathcal{M}_{B,h})$ of f is zero.*

The expression for $T^1(f, \mathcal{M}_{B,h})$ can be found for example in [Ols06a, Theorem 1.1]. Then using that $\mathbb{L}_{X/B} \cong \Omega_{X/B}$ in this case, one gets the following corollary.

Corollary 4.3. *A family $f : X \rightarrow B$ of canonically polarized manifolds over a compact base is rigid if $H^1(X, \mathcal{T}_{X/B}) = 0$.*

Lemma 4.4. *If $f : Y \rightarrow S \times T$ is a family of curves of genus at least two with S and T projective manifolds, such that for some $t \in T$, the restriction $W := Y_t \rightarrow S \times \{t\}$ is non-isotrivial, then there is a finite étale cover $U \rightarrow T$ from a variety, such that the following isomorphism holds*

$$(4.4.1) \quad \begin{array}{ccc} Y \times_{S \times T} S \times U & \cong & Y \times_T U \xleftarrow{\cong} W \times U \\ \downarrow & & \downarrow \\ S \times U & \xlongequal{\quad} & S \times U \end{array}$$

Proof. First, notice that

$$H^1(W, \mathcal{T}_{W/S}) \cong H^1(W, \omega_{W/S}^{-1}) = 0,$$

by [EV92, Corollary 5.12.c] and Corollary 3.4. Hence, by Corollary 4.3, $W \rightarrow S$ is rigid. That is, for any $t \in T$, $Y|_{S \times \{t\}} \cong W$ as schemes over S .

Let h be the Hilbert polynomial of $W \rightarrow S$. By Lemma 4.1 we know that $\mathcal{M}_{S,h}$ is DM stack of finite type. In particular it has an étale (not necessarily finite) cover $\pi : V \rightarrow \mathcal{M}_{S,h}$ by a scheme. Hence $\pi^{-1}([W \rightarrow S])$ is a zero dimensional scheme of finite type, which is then consequently proper. Moreover, all its subschemes are proper. The family $Y \rightarrow S \times T$ defines a map $T \rightarrow \mathcal{M}_{S,h}$ with zero dimensional image. Define $\tilde{U} := T \times_{\mathcal{M}_{S,h}} V$. Then \tilde{U} is a scheme, $\tilde{U} \rightarrow V$ is proper, and $\tilde{U} \rightarrow T$ is étale. By the second of these and the properness of all subschemes of $\pi^{-1}([W \rightarrow S])$, \tilde{U} is proper. Moreover, by the third one, \tilde{U} is the disjoint union of projective manifolds. Choose any of these, and define it to be U . Then U factorizes through a point of $\pi^{-1}([W \rightarrow S])$, which implies, that the associated family to $U \rightarrow \mathcal{M}_{S,h}$ is the trivial family $W \times U \rightarrow S \times U$. However, $U \rightarrow \mathcal{M}_{S,h}$ also factorizes through $T \rightarrow \mathcal{M}_{S,h}$, which gives us the isomorphism (4.4.1) \square

Lemma 4.5. *If $f : X \rightarrow B$ is a smooth map onto a smooth curve, then for the normal Stein-factorization*

$$\begin{array}{ccccc} & & f & & \\ & \nearrow & & \searrow & \\ X & \xrightarrow{g} & B' & \xrightarrow{h} & B, \end{array}$$

the following holds: B' is a smooth curve, g is smooth and h is étale.

Proof. One obtains smoothness of B' by the equivalence of normality and smoothness in dimension one. For the rest of the statements, take any point $P \in X$. Then there is a diagram of tangent maps

$$\begin{array}{ccccc} & & T_{f,P} & & \\ & \nearrow & & \searrow & \\ T_{X,P} & \xrightarrow{T_{g,P}} & T_{B',g(P)} & \xrightarrow{T_{h,g(P)}} & T_{B,f(P)}. \end{array}$$

Since the two tangent spaces on the right are one dimensional and the composition map is surjective, the only way to make the diagram commutative, if $T_{h,g(P)}$ is isomorphism and $T_{g,P}$ is surjective. This proves everything stated in the proposition. \square

Proof of Theorem 1.15. First we prove, that if all $\text{Var } f_i \geq 1$, then f is rigid. By Corollary 4.3, all we have to prove is that $H^1(X, \mathcal{T}_{X/B}) = 0$. Call g_i the maps $f_{i+1} \circ \dots \circ f_n : X \rightarrow X_i$. Then $\mathcal{T}_{X/B}$ has a filtration by the line bundles $g_i^* \mathcal{T}_{X_i/X_{i+1}}$. Hence it is enough to prove that $H^1(X, g_i^* \mathcal{T}_{X_i/X_{i+1}}) = 0$ for all i . This follows from Proposition 3.1, Corollary 3.4 and the vanishing theorem [EV92, Corollary 5.12.c].

We prove the other direction (or other statement) by induction on n . For $n = 1$ it is true by Lemma 4.4. So, assume that n is arbitrary, and the statement is true for $n - 1$. Then there are two possibilities. f_n is either isotrivial or not. If it is isotrivial, then let F be its fiber. By applying first Lemma 4.4 one gets the upper Cartesian square of the above diagram, and then Lemma 4.5 gives the lower factorization $X'_{n-1} \rightarrow B' \rightarrow B$.

$$\begin{array}{ccc} X & \xleftarrow{\quad} & F \times X'_{n-1} \\ \downarrow & & \downarrow \\ X_{n-1} & \xleftarrow{\text{étale}} & X'_{n-1} \\ \downarrow & & \downarrow \text{family of canonically polarized manifolds} \\ B & \xleftarrow{\text{étale, finite}} & B' \end{array}$$

As it is indicated on the diagram, $X'_{n-1} \rightarrow B'$ has canonically polarized fibers, since all its fibers are étale covers of fibers of $X_{n-1} \rightarrow B$. So, by

setting $W := X'_{n-1}$ and $Y := F$, the inductual step is proven if f_n is isotrivial.

If f_n is not isotrivial, then there must be some other i , for which f_i is isotrivial. However, then using the inductual hypothesis, there is a diagram as follows.

$$\begin{array}{ccc} X_{n-1} & \xleftarrow{\text{étale}} & X'_{n-1} := W_{n-1} \times Y_{n-1} \\ \downarrow & & \downarrow \\ B & \xleftarrow{\text{étale, finite}} & B'' \end{array}$$

Here Y_{n-1} is a canonically polarized manifold, the map $W_{n-1} \rightarrow B''$ is a family of canonically polarized manifolds and $W_{n-1} \times Y_{n-1} \rightarrow B''$ is the composition of the first projection with the map $W_{n-1} \rightarrow B''$. Define $X' := X \times_{X_{n-1}} X'_{n-1}$. Since $X \rightarrow X_{n-1}$ is not isotrivial, same holds for $X' \rightarrow X'_{n-1}$. Then, there is either a $w \in W_{n-1}$ or a $y \in Y_{n-1}$, such that $X' \rightarrow X'_{n-1}$ is non-isotrivial over $\{w\} \times Y_{n-1}$ or $W_{n-1} \times \{y\}$. Assume first, that the first case is happening. Define then $Y := X'|_{\{w\} \times Y}$. By Lemma 4.4, we obtain a Cartesian diagram as follows.

$$\begin{array}{ccc} X_{n-1} & \xleftarrow{\text{étale}} & W \times Y \\ \downarrow & & \downarrow \\ X'_{n-1} := W_{n-1} \times Y & \xleftarrow{\text{étale, finite}} & W \times Y_{n-1} \end{array}$$

Then by taking Stein factorization of $W \rightarrow B''$ and using Lemma 4.5, we get the following diagram, where the preceding construction is also included and which proves the inductual step if $X' \rightarrow X'_{n-1}$ was non-isotrivial over $\{w\} \times Y_{n-1}$.

$$\begin{array}{ccccc} X & \xleftarrow{\text{étale}} & X' & \xleftarrow{\text{étale}} & W \times Y \\ \downarrow & & \downarrow & & \downarrow \\ X_{n-1} & \xleftarrow{\quad} & X'_{n-1} = W_{n-1} \times Y_{n-1} & \xleftarrow{\quad} & W \times Y_{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ B & \xleftarrow{\text{étale, finite}} & B'' & \xleftarrow{\text{étale, finite}} & B' \end{array}$$

We conclude the proof with the case when $X' \rightarrow X'_{n-1}$ is non-isotrivial over $W_{n-1} \times \{y\}$. Then, define $W := X'|_{W_{n-1} \times \{y\}}$. Using Lemma 4.4

again yields the following diagram, where the left top square is Cartesian.

$$\begin{array}{ccccc}
 X & \longleftarrow & X' & \longleftarrow & W \times Y'_{n-1} \\
 \downarrow & & \downarrow & & \downarrow \\
 X_{n-1} & \longleftarrow & X'_{n-1} = W_{n-1} \times Y_{n-1} & \longleftarrow & W_{n-1} \times Y'_{n-1} \\
 \downarrow & & \downarrow & & \\
 B & \longleftarrow & B'' & &
 \end{array}$$

Setting $Y := Y'_{n-1}$ and $B' := B''$ yields the result in this case too. \square

5. ARBITRARY BASES

Here we treat the arbitrary base case. That is we allow B of Notation 1.9 to be affine too. The entire section is devoted to the proof of Theorem 1.17.

First we try to convey an intuition of why considering non-compact bases are much harder than the compact ones. The basic problem is that an entire class of new deformations appear if B is not compact. Intuitively the following happens. Consider a deformation of the tower in Notation 1.9, in the case of $n = 2$. Assume for simplicity that the deformation is such that the middle level deforms too. That is, we have a diagram of two Cartesian squares:

$$\begin{array}{ccc}
 X = X_2 & \longrightarrow & X' = X'_2 \\
 \downarrow f_2 & & \downarrow f'_2 \\
 X_1 & \longrightarrow & X'_1 \\
 \downarrow f_1 & & \downarrow f'_1 \\
 B & \longrightarrow & B \times T
 \end{array}
 \quad , \quad
 \begin{array}{c}
 \text{curved arrow } f \text{ from } X \text{ to } B \\
 \text{curved arrow } f' \text{ from } X' \text{ to } B \times T
 \end{array}$$

where T is a (not necessarily projective) smooth curve. We also assume that f' is smooth. If B was projective, then the smoothness of f_1 and f_2 and the open property of smoothness would imply that f'_1 and f'_2 are smooth too. However, as soon as we pass from non-compact to affine base, neither f'_1 nor f'_2 have any reason to be smooth. In fact, they are not smooth in general.

Being in a more subtle situation means, that the proof in this case will be based on a different method. In fact, we prove rigidity using the iterated Kodaira-Spencer map (Definition 2.4), as stated in Theorem 1.17. For the entire section we are in the situation of Notation 1.9. Since the statement of Theorem 1.17 is local, we assume that B is affine. To get rid of the indices,

we introduce $Y := X_1$, $g := f_2$, $h := f_1$. Hence we are in the situation

$$(5.0.1) \quad \begin{array}{ccccc} & & f & & \\ & \nearrow & & \searrow & \\ X & \xrightarrow{g} & Y & \xrightarrow{h} & B. \end{array}$$

Consider the following commutative diagram with exact rows.

$$(5.0.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I}_{X/B} \otimes g^* \mathcal{I}_{Y/B} & \longrightarrow & \mathcal{I}_X \otimes g^* \mathcal{I}_{Y/B} & \longrightarrow & f^* \mathcal{I}_B \otimes g^* \mathcal{I}_{Y/B} \longrightarrow 0, \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathcal{I}_{X/B} \otimes \mathcal{I}_{X/B} & \longrightarrow & \mathcal{I}_X \otimes \mathcal{I}_{X/B} & \longrightarrow & f^* \mathcal{I}_B \otimes \mathcal{I}_{X/B} \longrightarrow 0 \\ & & \downarrow \quad \curvearrowright & & \downarrow & & \parallel \\ 0 & \longrightarrow & \wedge^2 \mathcal{I}_{X/B} & \longrightarrow & \wedge^2 \mathcal{I}_X & \longrightarrow & f^* \mathcal{I}_B \otimes \mathcal{I}_{X/B} \longrightarrow 0 \end{array}$$

where

- the homomorphism $\mathcal{I}_{X/B} \otimes \mathcal{I}_{X/B} \rightarrow \wedge^2 \mathcal{I}_{X/B}$ is the wedge product map,
- the homomorphism $\mathcal{I}_X \otimes \mathcal{I}_{X/B} \rightarrow \wedge^2 \mathcal{I}_X$ is the embedding $\mathcal{I}_X \otimes \mathcal{I}_{X/B} \rightarrow \mathcal{I}_X \otimes \mathcal{I}_X$ composed with the wedge product $\mathcal{I}_X \otimes \mathcal{I}_X \rightarrow \wedge^2 \mathcal{I}_X$ and
- the homomorphism $\wedge^2 \mathcal{I}_{X/B} \rightarrow \mathcal{I}_{X/B} \otimes \mathcal{I}_{X/B}$ is the splitting of the wedge product map, given by $a \wedge b \mapsto \frac{1}{2}(a \otimes b - b \otimes a)$

Recall the homomorphisms ρ_i from Definition 2.4. Our aim is to show that $\rho_2 : R^1 f_*(f^* \mathcal{I}_B \otimes \mathcal{I}_{X/B}) \rightarrow R^2 f_*(\wedge^2 \mathcal{I}_{X/B})$ is injective on the image of $\rho_1 : \mathcal{I}_B^{\otimes 2} \rightarrow R^1 f_*(f^* \mathcal{I}_B \otimes \mathcal{I}_{X/B})$. Clearly, that will yield the injectivity of $\text{ks}_f = \rho_2 \circ \rho_1$.

NOTATION 5.1. Taking long exact sequences of derived pushforwards of the rows of (5.0.2) yields the following commutative diagram. We also introduce names for certain homomorphisms in the diagram.

$$\begin{array}{ccc} R^1 f_*(g^* \mathcal{I}_{Y/B} \otimes f^* \mathcal{I}_B) & \xrightarrow{\eta} & R^2 f_*(\mathcal{I}_{X/B} \otimes g^* \mathcal{I}_{Y/B}) \\ \delta \uparrow & & \uparrow \\ R^1 f_*(\mathcal{I}_{X/B} \otimes f^* \mathcal{I}_B) & \xrightarrow{\beta} & R^2 f_*(\mathcal{I}_{X/B} \otimes \mathcal{I}_{X/B}) \\ \parallel & & \downarrow \quad \curvearrowright \\ R^1 f_*(\mathcal{I}_{X/B} \otimes f^* \mathcal{I}_B) & \xrightarrow{\alpha} & R^2 f_*(\wedge^2 \mathcal{I}_{X/B}) \end{array}$$

Now we prove Theorem 1.17. In fact important parts are done in Propositions 5.2, 5.4 and 5.5, afterwards.

Proof of Theorem 1.17. We use Notation 5.1. By Proposition 5.2 and Proposition 5.4 both δ and η are generically injective. Hence, so is β . Consider now the following commutative diagram.

$$\begin{array}{ccccc}
 & & \nu & & \\
 & \nearrow & & \searrow & \\
 \mathcal{T}_B^{\otimes 2} & \xrightarrow{\text{ks}_f = \rho_1} & \mathcal{T}_B \otimes R^1 f_*(\mathcal{T}_{X/B}) & \xrightarrow{\beta} & R^2 f_*(\mathcal{T}_{X/B} \otimes \mathcal{T}_{X/B}) \\
 \parallel & & \parallel & & \downarrow \gamma \quad \varepsilon \\
 \mathcal{T}_B^{\otimes 2} & \xrightarrow{\text{ks}_f = \rho_1} & \mathcal{T}_B \otimes R^1 f_*(\mathcal{T}_{X/B}) & \longrightarrow & R^2 f_*(\wedge^2 \mathcal{T}_{X/B}) \\
 & \searrow & & \nearrow & \\
 & & \text{iks}_f & &
 \end{array}$$

Since h has variation 1, the same holds for f . One reason is for example that a variety has only finitely many dominant general type images up to birational equivalence (e.g., [HM06, Corollary 1.4]). Hence ks_f is injective. Then $\nu := \beta \circ \text{ks}_f$ is generically injective and also injective, since for homomorphisms from torsion free sheaves on varieties generic injectivity implies injectivity. By Proposition 5.5, $\text{im } \nu \subseteq \text{im } \varepsilon$. Since ε is a splitting of the surjection γ , this means, that γ maps $\text{im } \nu$ injectively. Hence $\text{iks}_f := \gamma \circ \nu$ is injective too. \square

The rest of the section deals with the propositions referenced by the proof of Theorem 1.17.

Proposition 5.2. *In the situation of Notation 5.1, δ is generically injective.*

Proof. Consider the following exact sequence.

$$(5.2.1) \quad 0 \longrightarrow \mathcal{T}_{X/Y} \longrightarrow \mathcal{T}_{X/B} \longrightarrow g^* \mathcal{T}_{Y/B} \longrightarrow 0$$

Since g has maximal variation, $X_b \rightarrow Y_b$ is non-isotrivial for generic $b \in B$. Hence for generic $b \in B$, ω_{X_b/Y_b} is ample by Corollary 3.3. Then by Kodaira vanishing $H^1(X_b, \mathcal{T}_{X/Y}) = 0$. So, $R_1 f_* \mathcal{T}_{X/Y}$ is torsion. Hence, taking the long exact sequence of derived pushforwards of (5.2.1) yields that the natural map

$$R^1 f_* \mathcal{T}_{X/B} \rightarrow R^1 f_* g^* \mathcal{T}_{Y/B}$$

is generically an injection. \square

For the next proposition we need a lemma first.

Lemma 5.3. *If \mathcal{E} is an ample vector bundle over a projective smooth curve, $\xi : \mathcal{E} \rightarrow \mathcal{H}$ a generically surjective homomorphism onto a vector bundle, then $\ker \xi \otimes \det \mathcal{H}$ is an ample vector bundle.*

Proof. Assume, there is a surjection $\phi : \ker \xi \otimes \det \mathcal{H} \rightarrow \mathcal{M}$ onto a line bundle. Then one can form the pushout diagram

$$(5.3.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \ker \xi \otimes \det \mathcal{H} & \longrightarrow & \mathcal{E} \otimes \det \mathcal{H} & \longrightarrow & \mathcal{H} \otimes \det \mathcal{H} \\ & & \downarrow \phi & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{H} \otimes \det \mathcal{H} \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0, & & \end{array}$$

where $\mathcal{F} := \mathcal{E} \otimes \det \mathcal{H} / \ker \phi$. Since \mathcal{E} is ample, so is $\mathcal{F} \otimes (\det \mathcal{H})^{-1}$. That is,

$$\begin{aligned} \deg \mathcal{F} &> (\operatorname{rk} \mathcal{F}) \deg(\det \mathcal{H}) = (\operatorname{rk} \mathcal{H} + 1) \deg(\det \mathcal{H}) \\ &= \deg(\mathcal{H} \otimes \det \mathcal{H}). \end{aligned}$$

This implies, by the bottom exact row of (5.3.1) that $\deg \mathcal{M} > 0$ (notice, that by construction the right most edge in that row is generically surjective). Hence all line bundle quotients of $\ker \xi \otimes \det \mathcal{H}$ have positive degree. If τ is a finite map of smooth curves, then the same holds for $\tau^*(\ker \xi \otimes \det \mathcal{H})$, since it is isomorphic to $(\ker(\tau^* \mathcal{E} \rightarrow \tau^* \mathcal{H})) \otimes \tau^* \det \mathcal{H}$ and $\tau^* \mathcal{E}$ is ample too. This shows that $\ker \xi \otimes \det \mathcal{H}$ is indeed ample. \square

Proposition 5.4. *In the situation of Notation 5.1, η is generically injective.*

Proof. To prove the generic injectivity of η we would need that

$$R^1 f_*(\mathcal{I}_X \otimes g^* \mathcal{I}_{Y/B})$$

is torsion. First, we show that

$$(5.4.1) \quad g_*(\mathcal{I}_X \otimes g^* \mathcal{I}_{Y/B}) \cong g_* \mathcal{I}_X \otimes \mathcal{I}_{Y/B} = 0.$$

Consider the following exact sequence.

$$0 \longrightarrow \mathcal{I}_{X/Y} \longrightarrow \mathcal{I}_X \longrightarrow g^* \mathcal{I}_Y \longrightarrow 0$$

Then by the pushforward long exact sequence we obtain

$$0 \longrightarrow g_* \mathcal{I}_{X/Y} \otimes \mathcal{I}_{Y/B} = 0 \longrightarrow g_* \mathcal{I}_X \otimes \mathcal{I}_{Y/B}$$

$$\longrightarrow \mathcal{I}_Y \otimes \mathcal{I}_{Y/B} \longrightarrow R^1 g_* \mathcal{I}_{X/Y} \otimes \mathcal{I}_{Y/B},$$

where the last map is ks_g tensored with $\mathcal{I}_{Y/B}$. Since g has maximal variation, this map is injective, which proves (5.4.1).

So, by the Grothendieck spectral sequence it is enough to show that

$$h_* R^1 g_*(\mathcal{I}_X \otimes g^* \mathcal{I}_{Y/B})$$

is torsion. By relative duality the following isomorphisms hold.

$$\begin{aligned} h_* R^1 g_*(\mathcal{T}_X \otimes g^* \mathcal{T}_{Y/B}) &\cong h_*((g_*(\omega_{X/Y} \otimes \Omega_X \otimes g^* \omega_{Y/B}))^*) \\ &\cong h_*((g_*(\omega_{X/B} \otimes \Omega_X))^*) \end{aligned}$$

So it is enough to show, that $h^0(Y_b, (g_*(\omega_{X/B} \otimes \Omega_X))^*) = 0$ for generic $b \in B$. In general $g_*(\omega_{X/B} \otimes \Omega_X)$ is not a vector bundle, however being the pushforward of a torsion free sheaf it is torsion free at least. Hence, since Y is smooth and $\dim Y = 2$, it is a vector bundle except at finitely many points. By leaving out the images of those points from B , we may assume, that $g_*(\omega_{X/B} \otimes \Omega_X)$ is in fact locally free.

Consider the following short exact sequence.

$$0 \longrightarrow \omega_{X/B} \otimes g^* \Omega_Y \longrightarrow \omega_{X/B} \otimes \Omega_X \longrightarrow \omega_{X/B} \otimes \omega_{X/Y} \longrightarrow 0$$

By pushing it forward one obtains

$$\begin{aligned} 0 \longrightarrow g_* \omega_{X/Y} \otimes \omega_{Y/B} \otimes \Omega_Y &\longrightarrow g_*(\omega_{X/B} \otimes \Omega_X) \\ &\longrightarrow g_*(\omega_{X/Y}^{\otimes 2}) \otimes \omega_{Y/B} \xrightarrow{\text{ks}_g} R^1 g_* \omega_{X/B} \otimes \Omega_Y \cong \omega_{Y/B} \otimes \Omega_Y, \end{aligned}$$

where the last homomorphism is the dual of ks_g tensored with $\omega_{Y/B}$. Again, im ks_g is not necessarily locally free, but being a subsheaf of a torsion free sheaf, it is torsion free. Hence as before, we may assume that it is in fact locally free. Then we see, that $g_*(\omega_{X/B} \otimes \Omega_X)$ is the extension of two locally free sheaves: $g_* \omega_{X/Y} \otimes \omega_{Y/B} \otimes \Omega_Y$ and $\ker \text{ks}_g$. We conclude our proof, by showing that

$$(5.4.2) \quad h^0(Y_b, (g_* \omega_{X/Y} \otimes \omega_{Y/B} \otimes \Omega_Y)^*) = 0$$

and

$$(5.4.3) \quad h^0(Y_b, (\ker \text{ks}_g)^*) = 0$$

for generic $b \in B$.

For the first one, since Ω_Y is the extension of two nef line bundles by Proposition 3.1, it is also nef. The pushforward $g_* \omega_{X/Y}$ is nef too (e.g., [Vie83, Theorem 4.1]) and $\omega_{Y/B}$ is h -ample. Hence $g_* \omega_{X/Y} \otimes \omega_{Y/B} \otimes \Omega_Y$ is ample on Y_b for each $b \in B$. This implies (5.4.2) for every $b \in B$.

To show (5.4.3), notice that ks_g is generically surjective by the assumption $\text{Var } g = 2$. By possibly restricting B we may assume that ks_g is generically surjective on each Y_b . Notice, that $h^0(X_y, \omega_{X_y}^{\otimes 2})$ is a constant function of y . Hence by [Har77, Corollary III.12.9], $g_*(\omega_{X/Y}^{\otimes 2})$ is locally free. Fix

$$0 \longrightarrow (\ker \mathrm{ks}_g)|_{Y_b} \longrightarrow g_*(\omega_{X/Y}^{\otimes 2}) \otimes \omega_{Y_b} \xrightarrow{\mathrm{ks}_g|_{Y_b}} \Omega_Y \otimes \omega_{Y_b} ,$$

Proposition 5.5. *The image of the composition*

$$\begin{array}{ccccc} & & \nu & & \\ & \curvearrowright & & \curvearrowleft & \\ \mathcal{I}_B^{\otimes 2} & \longrightarrow & \mathcal{I}_B \otimes R^1 f_*(\mathcal{I}_{X/B}) & \longrightarrow & R^2 f_*(\mathcal{I}_{X/B} \otimes \mathcal{I}_{X/B}) \end{array}$$

Since we assumed that B is affine, we can replace the derived pushforwards by global cohomology in Notation 5.1. Then we get the following diagram.

$$\begin{array}{ccc} H^0(X, f^* \mathcal{T}_B \otimes f^* \mathcal{T}_B) & \xrightarrow{\text{ks}_f} & H^1(X, f^* \mathcal{T}_B \otimes \mathcal{T}_{X/B}) \\ & \searrow \nu & \downarrow \beta \\ & & H^2(X, \mathcal{T}_{X/B} \otimes \mathcal{T}_{X/B}) \end{array}$$

$$(5.5.1) \quad c := \sum_{i=1}^3 c_i d\bar{z}_i \quad c_i \in \mathcal{T}_{X/B}$$
$$0 \longrightarrow f^* \mathcal{I}_B \otimes \mathcal{I}_{X/B} \longrightarrow f^* \mathcal{I}_B \otimes \mathcal{I}_X \longrightarrow f^* \mathcal{I}_B \otimes f^* \mathcal{I}_B \longrightarrow 0$$

So, we lift $a \otimes a$ to get $a \otimes b$ and then we apply $\bar{\partial}$. We obtain the following (remember, a is holomorphic, hence $\bar{\partial}(a) = 0$).

$$\nu(a \otimes a) = \bar{\partial}(a \otimes b) = a \otimes \bar{\partial}(b) = a \otimes c \in \mathcal{A}^{0,1}(\mathcal{T}_{X/B} \otimes f^* \mathcal{T}_B)$$

That is,

$$\nu(a \otimes a) = \beta(\text{ks}_f(a \otimes a)) = \beta(c \otimes a)$$

is obtained by feeding $a \otimes c$ to the edge morphism of the exact sequence

$$0 \longrightarrow \mathcal{T}_{X/B} \otimes \mathcal{T}_{X/B} \longrightarrow \mathcal{T}_X \otimes \mathcal{T}_{X/B} \longrightarrow f^* \mathcal{T}_B \otimes \mathcal{T}_{X/B} \longrightarrow 0$$

That is, we lift $a \otimes c$ to $b \otimes c$, and then we apply $\bar{\partial}$. We obtain the following (remember, c is Dolbeault-closed, hence $\bar{\partial}(c) = 0$).

$$\nu(a \otimes a) = \bar{\partial}(b \otimes c) = \bar{\partial}(b) \otimes c = c \otimes c \in \mathcal{A}^{0,2}(\mathcal{T}_{X/B} \otimes \mathcal{T}_{X/B})$$

We conclude our proof by showing that $\varepsilon(\gamma(c \otimes c)) = c \otimes c$. This part is slightly confusing, so we change to the local expression of (5.5.1).

$$\begin{aligned} \varepsilon(\gamma(c \otimes c)) &= \varepsilon \left(\gamma \left(\sum (c_i \otimes c_j) d\bar{z}_i \wedge d\bar{z}_j \right) \right) \\ &= \gamma \left(\sum (c_i \wedge c_j) d\bar{z}_i \wedge d\bar{z}_j \right) \\ &= \sum \frac{1}{2} (c_i \otimes c_j - c_j \otimes c_i) d\bar{z}_i \wedge d\bar{z}_j \\ &= \frac{1}{2} \sum_{i < j} (c_i \otimes c_j - c_j \otimes c_i) d\bar{z}_i \wedge d\bar{z}_j \\ &\quad + (c_j \otimes c_i - c_i \otimes c_j) d\bar{z}_j \wedge d\bar{z}_i \\ &= \sum_{i < j} (c_i \otimes c_j - c_j \otimes c_i) d\bar{z}_i \wedge d\bar{z}_j \\ &= \sum c_i \otimes c_j d\bar{z}_i \wedge d\bar{z}_j = c \otimes c \end{aligned}$$

□

6. CONNECTION TO TORELLI PROBLEM

In this section we present the connection between Theorem 1.17 and the Torelli problem. First, we list the required background. Then the proof of Theorem 1.20 will be straight forward. The given overview is deliberately short, since the Torelli problem is not the main objective of the paper. See for example [Voi02] or [CGGH83] for the details.

Given a family of projective manifolds $g : W \rightarrow Z$ and an integer w , the Torelli map is a holomorphic map from Z to some classifying space of Hodge structures with weight w , called the period domain. It sends $z \in Z$ to the point corresponding to the weight w Hodge structure of the w -th primitive cohomology of W_z . More precisely to give this map one also has

to fix a basepoint $z_0 \in Z$, and assign the change of Hodge structures relative to that point. With formulae

$$\begin{array}{c} z \in Z \\ \downarrow \phi_g \\ \left[\begin{array}{c} H^{w,0}(W_z, \mathbb{C})_0 \\ \oplus \\ \vdots \\ \oplus \\ H^{0,w}(W_z, \mathbb{C})_0 \end{array} \right] \subseteq H^w(W_z, \mathbb{C})_0 \cong H^w(W_{z_0}, \mathbb{C})_0 \bigg/ \pi_1(Z, z_0), \end{array}$$

where the 0 in lower index denotes the primitive cohomology and the quotient by $\pi_1(Z, z_0)$ means that ϕ_g in fact assigns an entire $\pi_1(Z, z_0)$ orbit, instead of just one point. We claim that Theorem 1.17 has consequences on ϕ_f , where f is the usual f introduced in Notation 1.9. The key object in this connection is infinitesimal variations of Hodge structures ([CGGH83]). We do not define these here, instead, we just quote one consequence of their theory, which we need. It is as follows.

Proposition 6.1. *For any smooth family $f : X \rightarrow B$ of relative dimension n over a smooth variety, the tangent map of the period map ϕ_f of weight n can be identified with*

$$T\phi_f : \mathcal{T}_B \rightarrow \bigoplus_{i=1}^n \text{Hom}(\mathcal{H}_0^{n-i+1, i-1}, \mathcal{H}_0^{n-i, i}),$$

where $\mathcal{H}_0^{n-i, i}$ is the $(n-i, i)$ primitive Hodge bundle. Let ϕ_f^i be ϕ_f composed with the i -th projection. Then there is a commutative diagram

$$(6.1.1) \quad \begin{array}{ccc} \mathcal{T}_B^{\otimes n} & \xrightarrow[\text{ks}_f]{} & R^n f_*(\wedge^n \mathcal{T}_{X/B}) \longrightarrow \text{Hom}^{\text{sym}}(\mathcal{H}^{n,0}, \mathcal{H}^{0,n}) \cong S^2(R^n f_* \mathcal{O}_X) \end{array}$$

where the homomorphism on the right is the dual of the natural product map $S^2(f_*(\omega_{X/B})) \rightarrow f_*(\omega_{X/B}^{\otimes 2})$, and the long arrow takes a point $b \in B$ to $\phi_f^n(b) \circ \dots \circ \phi_f^1(b)$ (which is a symmetric homomorphism by being obtained from a VHS).

Proof. Diagram [CGGH83, 2.a.15] gives us the statement for $B = \text{Spec } \mathbb{C}$. Since by [Har77, Theorem 12.8 and Corollary 12.9] there is base change for $R^i f_*$ generically on B , (6.1.1) commutes generically on B . However, since homomorphisms of vector bundles are determined uniquely on dense Zariski open sets, (6.1.1) commutes. \square

The next corollary is the immediate consequence of (6.1.1).

Corollary 6.2. *For any smooth family $f : X \rightarrow B$ of relative dimension n over a variety, if $S^2(f_*(\omega_{X/B})) \rightarrow f_*(\omega_{X/B}^{\otimes 2})$ is generically surjective and the iterated Kodaira-Spencer map iks_f is injective, then the tangent map $T\phi_f$ is injective.*

Proof of Theorem 1.20. Immediately follows from Corollary 6.2 and Theorem 1.17. \square

REMARK 6.3. The condition $S^2(H^0(X_b, \omega_{X_b})) \rightarrow H^0(X_b, \omega_{X_b}^{\otimes 2})$ being surjective at b , holds for example if ω_{X_b} is very ample. In the situation of Notation 1.9 for $n = 2$, this happens for example if $(X_1)_b$ is a canonical curve and $((f_2)_*\omega_{X/X_1})(-2P)|_{X_b}$ is an ample vector bundle for any $P \in X_b$. Since it is not in the main focus of the article we omit the proof of this very ampleness statement. However, it suggests that the surjectivity of $S^2(H^0(X_b, \omega_{X_b})) \rightarrow H^0(X_b, \omega_{X_b}^{\otimes 2})$ is the generic behavior. It seems that it should hold when X_b is a canonical curve and the Albanese variation of $X_b \rightarrow (X_1)_b$ is big. Notice also, that $S^2(H^0(X_b, \omega_{X_b})) \rightarrow H^0(X_b, \omega_{X_b}^{\otimes 2})$ is surjective, if ω_{X_b} is semi-ample, and the image of the map it determines is normal.

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